Pair dispersion in synthetic fully developed turbulence

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The Lagrangian statistics of relative dispersion in fully developed turbulence is numerically investigated. A scaling range spanning many decades is achieved by generating a two-dimensional velocity field by means of a stochastic process with prescribed statistics and of a dynamical model (shell model) with fluctuating characteristic times. When the velocity field obeys Kolmogorov similarity, the Lagrangian statistics is self similar and agrees with Richardson's predictions [Proc. R. Soc. London Ser. A **110**, 709 (1926)]. For intermittent velocity fields the scaling laws for the Lagrangian statistics are found to depend on the Eulerian intermittency in agreement with the multifractal description. As a consequence of the Kolmogorov law the Richardson law for the variance of pair separation is, however, not affected by intermittency corrections. Moreover, Lagrangian exponents do not depend on the particular Eulerian dynamics. A method of data analysis, based on fixed scale statistics rather than usual fixed time statistics, is shown to give much wider scaling range, and should be preferred for the analysis of experimental data. [S1063-651X(99)09112-6]

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I. INTRODUCTION

Understanding the statistics of particle pair dispersion in turbulent velocity fields is of great interest for both theoretical and practical implications. At variance with single particle dispersion, which depends mainly on large scale, energy containing eddies, pair dispersion is driven (at least at intermediate times) by velocity fluctuations at scales comparable with the pair separation. These small scale fluctuations are thought to be independent of the details of the large scale flow [1]. Since fully developed turbulence displays well known, nontrivial universal features in the Eulerian statistics of velocity differences [2,3], pair dispersion represents a starting point for the investigation of the general problem of the relationship between Eulerian and Lagrangian properties. Moreover, a deep comprehension of relative dispersion mechanisms is of fundamental importance from an applicative point of view, for a correct modelization of small scale diffusion and mixing properties.

Since the pioneering work by Richardson [4], many efforts have been done to confirm his law experimentally [2] or numerically [5–7]. Nevertheless, the main obstacle to a deep investigation of relative dispersion in turbulence remains the lack of sufficient statistics due to technical difficulties in laboratory experiments and to the moderate inertial range reached in direct numerical simulations.

In this paper we present a detailed investigation of the statistics of relative dispersion from extensive direct numerical simulations of particle pairs advected by two-dimensional synthetic turbulent velocity fields with prescribed Eulerian statistical features. First we consider the probability distribution of Lagrangian quantities in a self similar Kolmogorov-like flow, and confirm the Richardson-Obukhov predictions. We then investigate the effects of Eulerian intermittency on Lagrangian statistics. We find deviations of the scaling exponents from Richardson's values, i.e., ''Lagrangian intermittency.'' These effects cannot be captured by dimensional arguments alone. The simplest step beyond dimensional analysis is the extension of the multifractal description successfully used for Eulerian statistics to Lagrangian quantities. Our numerical simulations agree with the predictions of the Lagrangian multifractal description.

To see possible effects of fluctuating characteristic times on the Lagrangian statistics, we also consider a velocity field generated by a shell model of turbulence. We find that relative dispersion scaling laws are not sensitive to the details of the Eulerian dynamics.

The intermittency corrections to relative dispersion are however small. Moreover, they can be hidden by the finite scaling range. Huge Reynolds numbers are necessary in order to clearly resolve the scaling exponents. To partially overcome these difficulties, we propose a methodology for the analysis of relative dispersion data based on Lagrangian statistics at fixed spatial particle separation. In particular, the statistics of ''doubling times'' — the time that two particles spend to double their separation — seems a very promising tool in data analysis.

In Sec. II we address the problem of relative dispersion in fully developed, homogeneous, and isotropic turbulence. In Sec. III we propose a method to construct intermittent Eulerian velocity fields with prescribed intermittent properties. In Sec. IV results on the Lagrangian statistics of particle pairs advected by a nonintermittent Kolmogorov-like velocity field are presented. In Sec. V the method of doubling times is introduced, and its advantages over the usual fixed time statistics are discussed. In Sec. VI we consider intermittent Eu-

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lerian velocity fields. The effects on the Lagrangian statistics are discussed in a multifractal framework and compared with numerical data. In Sec. VII conclusions are drawn.

II. RICHARDSON LAW

We consider the dispersion of pairs of particles passively advected by an homogeneous, isotropic, fully developed turbulent field. Due to the incompressibility of the velocity field the particles will, on average, separate one from another [8,9]. The statistics of particle pair separation is conveniently summarized by the probability density function $p(\boldsymbol{R},t)$ of the distance between pairs of particles at a given time, called the *distance neighbor function* by Richardson [4].

In view of the diffusive effect exerted by the turbulent motion on the advected particles, Richardson argued that the time evolution of the distance neighbor function could be described by a proper diffusion equation

$$\frac{\partial p(\boldsymbol{R},t)}{\partial t}=\frac{\partial}{\partial R_j}\left(K(R)\frac{\partial p(\boldsymbol{R},t)}{\partial R_j}\right), \tag{1}$$

with an R dependent scalar turbulent diffusivity $K(R)$. From a collection of experimental data, Richardson was able to obtain his celebrated ''4/3'' law

$$K(R)=\alpha R^{4/3}, \tag{2}$$

where α is a constant. This choice for the diffusivity relied mainly on empirical grounds: dependence of the vertical eddy diffusivity in the atmosphere with the altitude.

In three dimensions the solution of Eq. (1) is

$$p(\boldsymbol{R},t)=\mathcal{N}(\alpha t)^{-9/2}\exp\left(-\frac{9R^{2/3}}{4\alpha t}\right), \tag{3}$$

where $\mathcal{N}$ is a normalization factor, which immediately leads to the growth laws for the moments of particle separation:

$$\langle R^{2n}(t)\rangle=\int dR R^{2n}p(\boldsymbol{R},t)\sim t^{3n}. \tag{4}$$

Scaling (4) can also be derived by a simple dimensional argument due to Obukhov [2], which uses the Kolmogorov similarity law for the Eulerian velocity increments over a distance R in fully developed turbulence:

$$\langle|\delta\boldsymbol{v}^{(E)}(\boldsymbol{R})|\rangle=\langle|\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{R})-\boldsymbol{v}(\boldsymbol{x})|\rangle\sim R^{1/3}, \tag{5}$$

with $R=|\boldsymbol{R}|$, and the particle pair separation equation

$$\frac{d\boldsymbol{R}}{dt}=\delta\boldsymbol{v}^{(L)}(\boldsymbol{R}), \tag{6}$$

where $\delta\boldsymbol{v}^{(L)}$ represents the velocity difference evaluated along the Lagrangian trajectory. Assuming that $\delta v^{(L)}(\boldsymbol{R})$ has the same scaling exponent of $|\delta\boldsymbol{v}^{(E)}(\boldsymbol{R})|$, from Eqs. (5) and (6) one obtains $dR^2/dt\sim R\,\delta v^{(L)}(R)\sim R^{4/3}$ and hence the Richardson's law $\langle R^2\rangle\sim t^3$ [cf. Eq. (4)]. The assumption that the Lagrangian velocity difference has the same Kolmogorov scaling as the Eulerian one relies on the intuitive idea that the main contribution to the separation rate follows from eddies with a size comparable to the separation itself.

We remark that prediction (4), being based on dimensional grounds, can be obtained starting from different assumptions. On the other hand, the Richardson form [Eq. (3)] for the distance neighbor function depends on the details of the argument, and in particular on the validity of position (2). This is not the only possible form compatible with the known experimental data; indeed, in 1952, Batchelor proposed an alternative approach to the problem which led to a different (Gaussian) $p(\boldsymbol{R},t)$ [1]. As far as we know there is still no clear evidence supporting one or another form of the distance neighbor function of particle pairs. The most recent experimental results [10] support non-Gaussian tails close to the Richardson proposal.

At this point two remarks are in order. First, the determination of the specific functional dependencies — such as the shape of the distance-neighbor function $p(\boldsymbol{R},t)$ — lies beyond the possibilities of similarity hypotheses, and thus calls for additional hypotheses, like Eqs. (1) and (2). Second, diffusive approximations cannot describe intermittency effects on Lagrangian statistics. We thus have to deal with supplementary hypotheses which have a considerable degree of arbitrariness, and whose content can be mainly judged *a posteriori*. In this respect the use of synthetic velocity fields represents a flexible framework for a detailed analysis of pair dispersion statistics.

III. SYNTHETIC TURBULENT FIELD

The generation of a synthetic turbulent field which reproduces the relevant statistical features of fully developed turbulence is not an easy task. To obtain a physically sensible evolution for the velocity field, the fact that each eddy is subject to the action of all other eddies must be consistently included. Actually the overall effect amounts to only two main contributions: the sweeping exerted by larger eddies and the shearing due to eddies of comparable size. This is a substantial simplification; nevertheless the problem of properly mimicking the sweeping effects is still unsolved. It is relatively easy to construct a spatial, time independent, self affine, or multiaffine velocity field in any dimension. Then to let the particle separate some time evolution, at least in two dimensions, has to be introduced. The time evolution of the velocity field should be consistent with the field itself, i.e., the eddies should move with the local velocity itself. This is the essence of the sweeping problem which has been addressed in different ways in previous works [6,7].

In the study of particle pairs dispersion, and only in this case, we can overcome the sweeping problem by using *quasi-Lagrangian* (QL) coordinates [11], i.e., a reference frame attached to a fluid particle $\boldsymbol{r}_1(t)$. This choice bypasses the problem of sweeping, since relative velocities are unaffected by large scale advection, making the generation of realistic synthetic velocity fields simpler. The price to pay is that only the problem of two-particle dispersion can be well managed within this framework. The properties of single-particle Lagrangian statistics, for example, cannot be consistently treated. Thus our method cannot deal with nonhomogeneous situations where the velocity statistics depend on the position of the reference particle.

In the QL reference the first particle sits at the origin and is, by definition, at rest. The second particle is placed at distance (separation) $\boldsymbol{R}$ and is advected by the QL velocity

$$\boldsymbol{v}^{(QL)}(\boldsymbol{R},t)=\boldsymbol{v}(\boldsymbol{r}_1(t)+\boldsymbol{R},t)-\boldsymbol{v}(\boldsymbol{r}_1(t),t) \tag{7}$$

where $\boldsymbol{v}$ is the Eulerian velocity field and $\boldsymbol{r}_1(t)$ the trajectory in the original reference frame of the particle placed at the origin of the QL reference:

$$\frac{d}{dt}\boldsymbol{r}_1(t)=\boldsymbol{v}(\boldsymbol{r}_1(t),t). \tag{8}$$

By this change of coordinates the problem of pair dispersion in a Eulerian velocity field $\boldsymbol{v}$ is reduced to the problem of single particle dispersion in the QL velocity field $\boldsymbol{v}^{(QL)}(\boldsymbol{r},t)$. It is easy to show that whenever one considers homogeneous flows [11], the QL velocity differences have the same one-time statistical properties of the Eulerian ones. Indeed, one has

$$\begin{aligned}\delta\boldsymbol{v}^{(QL)}(\ell,t;\boldsymbol{r})&\equiv\boldsymbol{v}^{(QL)}(\boldsymbol{r}+\ell,t)-\boldsymbol{v}^{(QL)}(\boldsymbol{r},t)\\&=\boldsymbol{v}(\boldsymbol{r}_1(t)+\boldsymbol{r}+\ell,t)-\boldsymbol{v}(\boldsymbol{r}_1(t)+\boldsymbol{r},t)\\&\equiv\delta\boldsymbol{v}^{(E)}(\ell,t;\boldsymbol{r}_1(t)+\boldsymbol{r}).\end{aligned} \tag{9}$$

Thus, assuming homogeneity and averaging over the reference trajectory, for the structure functions one finds

$$\langle(\delta\boldsymbol{v}^{(QL)}(\ell,t))^p\rangle\sim\langle(\delta\boldsymbol{v}^{(E)}(\ell,t))^p\rangle\sim\ell^{\zeta_p}, \tag{10}$$

where the exponent ζ_p is a convex function of p, and $\zeta_3=1$.

When generating a synthetic velocity field for the particle pair dispersion we can fulfill a weaker condition requiring scaling (10) to be satisfied *only* along the line joining the two particles, since only moments of $\boldsymbol{v}^{(QL)}$ along this direction enter into the dynamics. This is a further substantial simplification: it is sufficient to build a QL velocity field with the proper scaling in the radial direction only. Needless to say, for three-particles dispersion a field with proper scaling in all directions must already be constructed. In the following we will consider only QL velocity fields, and the superscript will be omitted.

We consider only the two-dimensional case, where we can introduce a stream function for the QL velocity:

$$\boldsymbol{v}(\boldsymbol{r},t)=\boldsymbol{\nabla}\times\psi(\boldsymbol{r},t). \tag{11}$$

The extension to three-dimensional velocity fields does not present technical difficulties, but it is only more expensive in terms of numerical resources. Under isotropic conditions, the stream function can be decomposed in radial octaves as

$$\psi(\boldsymbol{r},\theta,t)=\sum_{i=1}^{N}\sum_{j=1}^{n}\frac{\phi_{i,j}(t)}{k_i}F(k_i r)G_{i,j}(\theta),\quad r=|\boldsymbol{r}|, \tag{12}$$

where $k_i=2^i$. Following a heuristic argument, one expects that for a given r sum (12) is essentially dominated by the term with i such that $r\sim2^{-i}$. This locality of contributions suggests a simple choice for the functional dependencies of the "basis functions":

$$F(x)=x^2(1-x)\quad\text{for}\quad 0\leqslant x\leqslant 1, \tag{13}$$

TABLE I. Theoretical and numerical fitted scaling exponent for simulations with an intermittent velocity field. The number of shells is $N=30$, corresponding to an integral Reynolds number Re $\simeq10^{10}$. ζ_p: Eulerian structure functions theoretical scaling exponents. ζ_p^{num}: Lagrangian structure function numerical scaling exponents. α_p: theoretical relative dispersion scaling exponents. α_p^{num}: numerical relative dispersion scaling exponents. ζ_p-p: theoretical doubling time scaling exponents. β_p^{num}: numerical doubling time scaling exponents. ζ_p^{SM}: Eulerian scaling exponents for the shell model.

p	ζ_p	ζ_p^{num}	α_p	α_p^{num}	ζ_p-p	β_p^{num}	ζ_p^{SM}
1	0.390	0.39	1.59	1.56	−0.610	−0.62	0.39
2	0.719	0.74	3.00	2.94	−1.281	−1.28	0.73
3	1.0	1.04	4.32	4.27	−2.0	−1.99	1.01
4	1.245	1.30	5.58	5.58	−2.755	−2.73	1.26
5	1.461	1.54	6.80	6.88	−3.539	−3.49	1.49
6	1.655	1.74	7.99	8.17	−4.345	−4.23	1.71

and zero otherwise. Moreover we take

$$G_{i,1}(\theta)=1,\quad G_{i,2}(\theta)=\cos(2\theta+\varphi_i), \tag{14}$$

and $G_{i,j}=0$ for $j>2$, with φ_i quenched random phases. This choice is rather general because it can be derived from the lowest order expansion for small r of a generic streamfunction in QL coordinates.

Under the usual locality conditions for infrared convergence, $\zeta_p<p$ [12], the leading contribution to the pth order structure function follows from the term in sum (12) with $r\simeq2^{-M}$, and is given by $\langle|\phi_{M,2}|^p\rangle$. Thus if $\phi_{i,j}(t)$ are stochastic processes with characteristic times $\tau_i\sim2^{-2i/3}$, to ensure the correct turnover time, zero mean and $\langle|\phi_{i,j}|^p\rangle\sim k_i^{-\zeta_p}$, scaling (10) will be accomplished.

An efficient way of to generate $\phi_{i,j}$ is [13]

$$\phi_{i,j}(t)=g_{i,j}(t)z_{1,j}(t)z_{2,j}(t)\cdots z_{i,j}(t), \tag{15}$$

where $z_{i,j}$ are independent, positive definite, identically distributed random processes with a characteristic time $\tau_i=2^{-2i/3}$, while $g_{i,j}$ are independent random processes with zero mean, $\langle g_{i,j}^2\rangle\sim k_i^{-2/3}$, and characteristic time τ_i.

The scaling exponents ζ_p are determined by the probability distribution of $z_{i,j}$ via

$$\zeta_p=\frac{p}{3}-\ln_2\langle z^p\rangle. \tag{16}$$

The Kolmogorov scaling $\zeta_p=p/3$ is recovered by fixing $z_{i,j}=1$. In the following we shall report results for both synthetic turbulent velocity fields with Kolmogorov scaling $\zeta_p=p/3$, and fields whose intermittency corrections to the Kolmogorov scaling are close to the experimental values for three dimensional turbulence [14]; see Table I.

IV. LAGRANGIAN STATISTICS IN THE ABSENCE OF EULERIAN INTERMITTENCY

When the advecting velocity field has Kolmogorov scaling $\zeta_p=p/3$, one expects Richardson's law $\langle R^2\rangle\sim t^3$ to hold. This is indeed very well verified in our numerical simula-

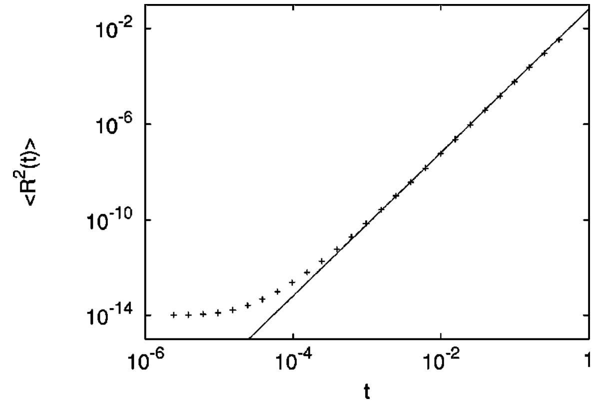

FIG. 1. Average variance of pair separation $\langle R^2(t)\rangle$ for simulation with $N=30$ octaves averaged over 10^4 realizations. The continuous line represents the Richardson scaling t^3.

tions over a wide range of scales, as shown in Fig. 1. The Reynolds number for the synthetic Eulerian velocity field can be defined as $\mathrm{Re}=(k_N/k_1)^{4/3}$, so that for this simulation with $N=30$ we have $\mathrm{Re}\simeq 10^{12}$.

From similarity arguments the distance neighbor function $p(\boldsymbol{R},t)$ in two dimensions should assume the self similar isotropic form

$$p(\boldsymbol{R},t)=t^{-3}\Phi(R/t^{3/2}), \tag{17}$$

where $\Phi(\xi)$ is a universal function whose shape cannot be predicted from similarity hypotheses. We checked the validity of Eq. (17) by rescaling the numerical distance neighbor functions with the theoretical average separation $\langle R\rangle\sim t^{3/2}$. The different curves collapse onto a unique curve $\Phi(R/t^{3/2})$; see Fig. 2. The collapse indicates that the process is indeed self similar in time. The continuous line in this figure is the Richardson prediction $\Phi(\xi)\propto\exp(-b\xi^{2/3})$, [cf. Eq. (3)], and is in good agreement with the numerical data. The distance neighbor functions obtained in our simulations clearly deviates from the Gaussian proposal of Batchelor [1] (see Fig. 2), and give strong support to the original Richardson prediction. Recent experimental results [10], although affected by larger uncertainty, support a similar conclusion.

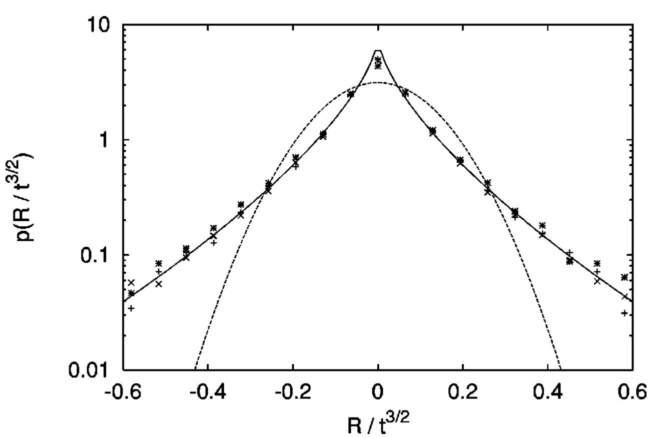

FIG. 2. Rescaled probability distribution functions of separations as function of $R/t^{3/2}$ for the simulation of Fig. 1 and $t=10^{-3}(*)$, $t=10^{-2}(\times)$, and $t=0.25(+)$. The continuous line represents the Richardson distribution (3), and the dashed line is the Gaussian distribution.

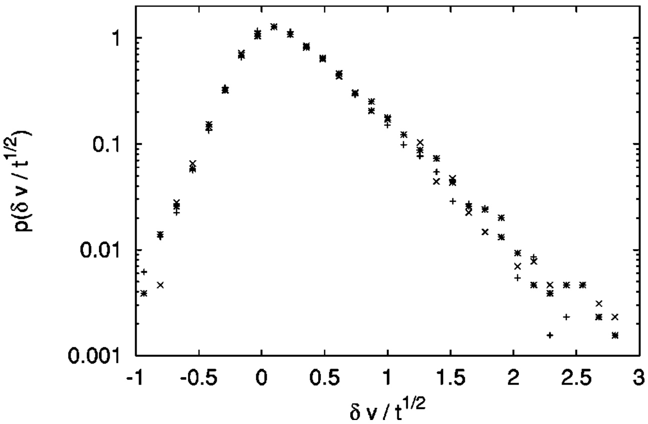

FIG. 3. Rescaled probability distribution functions of Lagrangian velocity differences (rescaled with $t^{1/2}$) for the same values of t as in Fig. 2.

Another interesting statistical quantity which can be investigated is the probability distribution function $p_L(\delta v|t)$ of Lagrangian velocity differences δv evaluated along the trajectory. Dimensional arguments lead to $\langle\delta v^2\rangle\sim t$, showing the accelerating nature of Richardson dispersion [2]. In Fig. 3 we plot the computed $p_L(\delta v|t)$ as function of $\delta v/t^{1/2}$. The collapse of curves for different times t demonstrates the validity of the scaling assumption.

V. DOUBLING TIME STATISTICS

A closer look at Fig. 1 shows that the power-law scaling regime $\langle R^2(t)\rangle\sim t^3$ is observed only well inside the inertial range: the scaling range for relative dispersion is reduced with respect to the Eulerian inertial range. To understand this effect consider a series of particle pair dispersion experiments, in which a couple of particles is released at time $t=0$ with initial separation R_0. At a fixed time t_1, as is usually done, we perform an average over all different experiments and compute $\langle R^2(t_1)\rangle$. It is clear that, unless t_1 is large enough that all particle pairs have ''forgotten'' their initial conditions, our average will be biased. This is at the origin of the flattening of $\langle R^2(t)\rangle$ for small times, which we can call a crossover from an initial condition dominated regime to a self similar regime. A similar effect is observed for times of the order of the integral time scale, since some particle pairs might have reached a separation larger than the integral scale and thus diffused normally, biasing the average, so that the curve $\langle R^2(t)\rangle$ flattened again. This effect is particularly evident for low Reynolds numbers, as shown in Fig. 4 for a simulation with $\mathrm{Re}\simeq 10^8$. This correction to a pure power law is far from being negligible, especially in experimental and direct numerical simulation data where the inertial range is generally limited by low Reynolds numbers and/or experimental apparatus. For example, Refs. [7,15] show quite clearly the difficulties that may arise in numerical simulations with the standard approach.

To overcome this difficulty we propose an alternative approach based on statistics at a fixed spatial scale. The method is in the spirit of a recently introduced generalization of the Lyapunov exponent to finite size perturbation (finite size

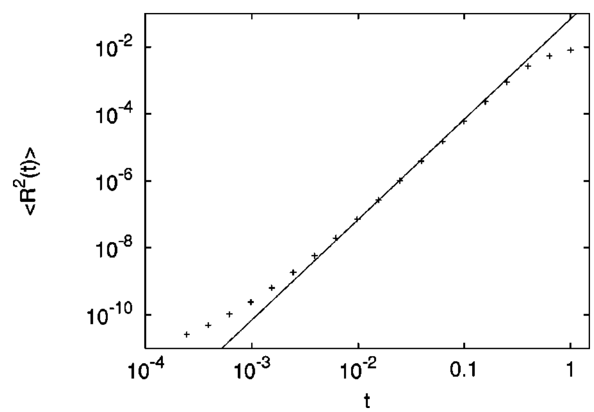

FIG. 4. Relative dispersion $\langle R^2(t)\rangle$ for $N=20$ octaves simulation averaged over 10^4 realizations. The line is the theoretical Richardson scaling t^3.

Lyapunov exponent) which has been successfully applied in the predictability problem [16] and in the diffusion problem [17]. Given a set of thresholds $R_n=2^nR_0$ within the inertial range, we compute the ''doubling time'' $T(R_n)$ defined as the time it takes for the particle pair separation to grow from a threshold R_n to the next one R_{n+1}. Averages are then performed over many dispersion experiments, i.e., particle pairs. The outstanding advantage of this kind of averaging at fixed scale separation, as opposite to a fixed time, is that it removes crossover effects since all sampled particle pairs belong to the inertial range.

The scaling properties of the doubling time statistics are obtained by simple dimensional arguments. The time it takes for the particle pair separation to grow from R to $2R$ can be dimensionally estimated as $T(R)\sim R/\delta v(R)$; for the inverse doubling times we thus expect the scaling

$$\left\langle \frac{1}{T^p(R)}\right\rangle \sim \frac{\langle \delta v(R)^p\rangle}{R^p} \sim R^{-2p/3}. \tag{18}$$

In Fig. 5 the great enhancement in the scaling range achieved by using ''doubling times'' is clearly evident.

The conclusion that can be drawn from this simple example is that the doubling time statistics allow for better estimations of the scaling exponents than the usual fixed time

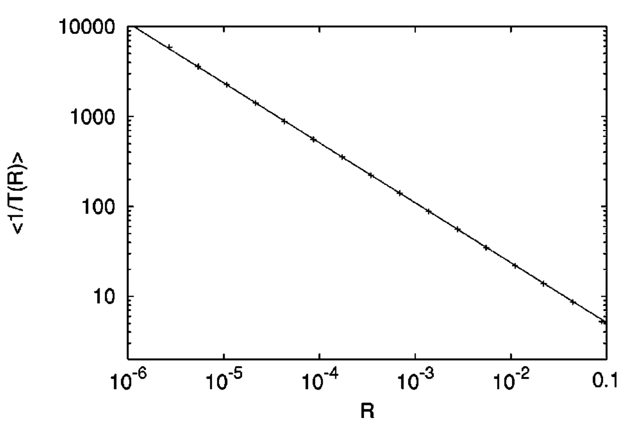

FIG. 5. Average inverse doubling time $\langle 1/T(R)\rangle$ for the same simulation as in Fig. 4. Observe the enhanced scaling region. The line is the theoretical Richardson scaling $R^{-2/3}$ [Eq. (18)].

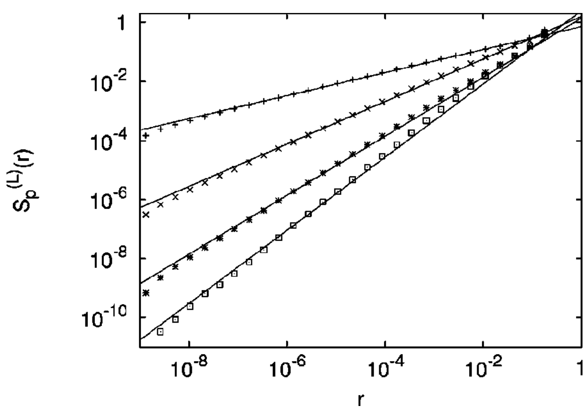

FIG. 6. Longitudinal Lagrangian structure functions $S_p^{(L)}(R)$ for $p=1,2,3,$ and 4 (from top to bottom) for an intermittent velocity field of $N=30$ octaves. The average is over 10^5 particle pairs. The continuous lines represents the theoretical scaling with exponents ζ_p given in Table I.

statistics. This property will be used in Sec. VI to investigate the scaling laws of relative dispersion in the presence of Eulerian intermittency.

VI. EFFECT OF EULERIAN INTERMITTENCY

In previous literature there have been very few attempts to investigate possible corrections to Lagrangian statistics stemming from Eulerian intermittency [18–21]. This is quite surprising especially if compared with the enormous amount of literature concerning the intermittency corrections to Eulerian statistics [3,22]. This mismatch is partly explained by the difficulty of having experimental checks for the proposed theoretical corrections. The use of synthetic velocity fields provides a first benchmark which is extremely easy and less expensive than experiments and direct numerical simulations.

In Fig. 6 we report the Lagrangian structure functions $S_p^{(L)}(r)=\langle(\delta v^{(L)}(r))^p\rangle$ computed recording the Lagrangian velocity difference whenever the particle pair separation equals r. The average is over many different particle pairs. The values of ζ_p for the intermittent Eulerian synthetic velocity field are reported in Table I. Observe the wide inertial range, over more than ten decades, corresponding to an integral Reynolds number of $\mathrm{Re}\simeq 10^{12}$. As opposed the Eulerian structure function, the first order Lagrangian structure function is nonzero, reflecting the average growth of particle pair separation.

The most interesting and nontrivial result is that scaling exponents for the Lagrangian structure functions have values very close to those of the Eulerian ζ_p (Table I). In terms of multifractal formalism [3,23], this result is restated by saying that the fractal dimensions $D(h)$ for the Lagrangian and Eulerian velocity statistics are the same.

With this result we can extend the dimensional arguments leading to Richardson's law to the intermittent case by using the multifractal formalism. From the definition

$$\frac{d}{dt}\langle R^p\rangle = p\langle R^{p-1}\delta v^{(L)}\rangle, \tag{19}$$

and using the multifractal representation for the velocity differences, we can write

$$\frac{d}{dt}\langle R^p\rangle \sim \int dh R^{p-1+h+3-D(h)}. \tag{20}$$

The time needed for the particle pair separation to reach the scale R is dominated by the largest time in the process, and can be dimensionally estimated as $t \sim R/\delta v^{(L)} \sim R^{1-h}$, leading to

$$\frac{d}{dt}\langle R^p\rangle \sim \int dh\, t^{[p+2+h-D(h)]/(1-h)}. \tag{21}$$

The integral is evaluated by saddle point method, and gives the final result $\langle R^p\rangle \sim t^{\alpha_p}$, with the scaling exponents

$$\alpha_p = \inf_h \left[\frac{p+3-D(h)}{1-h}\right]. \tag{22}$$

In the case of an intermittent Eulerian velocity field, the relative dispersion displays a nonlinear scaling exponent α_p (see Table I). However, there is an interesting result, already obtained in Ref. [18]. From the general multifractal formalism it follows that $3-D(h) \geqslant 1-3h$, where the equality is satisfied for the scaling exponent h_3 which realizes the third order structure function $\zeta_3=1$. As a consequence it follows that $\alpha_2=3$, and the Richardson's law $\langle R^2\rangle \sim t^3$ is not affected by intermittency corrections, while the other moments in general are. We note that the argument presented here leading to Eq. (22) is just one reasonable dimensional assumption which can be justified only *a posteriori* by numerical simulations or experimental data. However other assumptions are possible [18–20], leading, in general, to different predictions.

The scaling exponents α_p satisfy the inequality $\alpha_p/p < 3/2$ for $p>2$. This amounts to saying that, as time goes on, the right tail of the particle pair separation probability distribution function becomes less and less broad. In other words, due to the Eulerian intermittency, particle pairs are more likely to stay close to each other than to experience a large separation.

In Fig. 7 we show the moments $\langle R^p(t)\rangle$ for different p's. We find that $\langle R^2(t)\rangle$ displays a clear t^3 scaling law. The scaling region reduces as the moment increases, making the determination of the exponents α_p rather difficult. To overcome this difficulty we plot the moments $\langle R^p(t)\rangle$ compensated for by $\langle R^2(t)\rangle^{\alpha_p/3}$, which should result to be constant according to Eq. (22). For comparison we also plot the moment $p=4$ compensated for by normal scaling, i.e., $\langle R^4(t)\rangle \sim \langle R^2(t)\rangle^2$. It is evident that prediction (22) is compatible with our numerical data, while the Richardson scaling (4) is not. To be more quantitative, in Table I we report the values of α_p obtained from a best fit of $\langle R^p(t)\rangle$. The numerical values, although affected by the large uncertainty, are in good agreement with the theoretical predictions.

The time doubling analysis discussed in Sec. V proves to be very useful in the case of Eulerian intermittency. To see how the scaling of the doubling times is affected we can use the dimensional estimate for the doubling time $T(R) \sim R/\delta v(R)$. Fluctuations of the velocity fields induce fluctuations in the doubling time. Averaging over many realizations, we can write

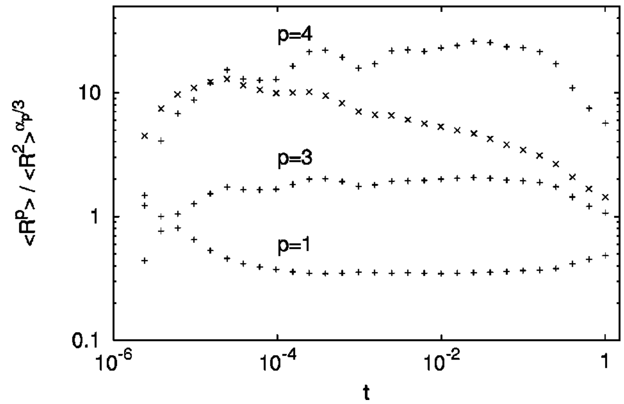

FIG. 7. Relative dispersion $\langle R^p(t)\rangle$ rescaled with $\langle R^2(t)\rangle^{\alpha_p/3}$ for $p=1$,3, and 4 (+). The almost constant plateau indicates a relative scaling in agreement with prediction (22). For comparison, we also plot $\langle R^4(t)\rangle$ rescaled with the nonintermittent prediction $\langle R^2(t)\rangle^2(\times)$, clearly indicating a deviation from normal scaling.

$$\left\langle \frac{1}{T^p(R)}\right\rangle \sim \int dh R^{p(h-1)} R^{3-D(h)} \sim R^{\zeta_p-p}. \tag{23}$$

The doubling time statistics contains, therefore, the same information as the relative dispersion exponents (22). Let us remark that in this case the exponent unaffected by Eulerian intermittency is $\zeta_3-3=-2$.

As reported in Fig. 8, our predictions are very well confirmed by numerical simulations. The plot of the compensated inverse time statistics clearly discriminates between the multiaffine scaling [Eq. (23)] and the affine scaling [Eq. (18)] (here reported only for $p=4$). Note also that the scaling region for the inverse time statistics (fixed scale statistics) is wider than that of Fig. 7 (fixed time statistics), so that the scaling exponent can be determined with higher accuracy. In Table. I we report the theoretical exponent $\beta_p=\zeta_p-p$ of Eq. (23) compared with the numerical values from best fit. The agreement is within 2%.

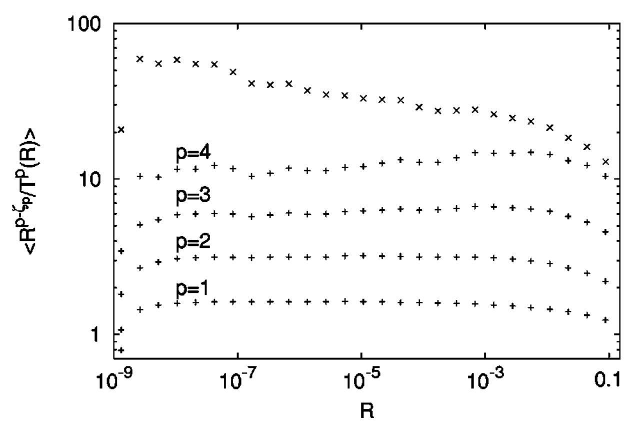

FIG. 8. Inverse doubling time statistics $\langle 1/T^p(R)\rangle$ compensated for by the multifractal prediction (18) R^{ζ_p-p} for $p=1,2,3$, and 4 (+). Inverse doubling times $\langle 1/T^4(R)\rangle$ compensated for by the nonintermittent prediction $R^{-8/3}(\times)$.

Fluctuating characteristic times

It must be pointed out that the construction of the synthetic field proposed here shows an inconsistency with the expected statistical properties imposed by Navier-Stokes equations. Indeed, the characteristic turnover time of eddies of size R is taken to be equal to $\tau(R)\sim R^{2/3}$. However, since dimensionally $\tau(R)\sim R/\delta v(R)$, as long as $\delta v(R)$ are fluctuating quantities the characteristic times must fluctuate as well. This effect is definitely enhanced in presence of intermittency, where the average characteristic times do not show simple scaling [11]. This finding could shed some doubt on the validity of the results obtained up to here.

We thus turn to popular dynamical models of turbulence, the so called shell models [22]. These are deterministic models which display a dynamic energy cascade toward small scales. The model is built in terms of shell variables u_n which represent the velocity differences in a wave number octave $k_n=k_0 2^n$ in quasi-Lagrangian coordinates. With a suitable choice of parameters, the model develops a chaotic dynamics which is responsible for the intermittency correction to the Kolmogorov scaling exponents remarkably close to the experimental data [22]. Being a complete deterministic system, the shell model also displays dynamic eddy turnover times with the same statistics expected for Navier-Stokes turbulence [24].

The particular model we use is a recently proposed shell model [25] for the complex variables u_n,

$$\frac{du_n}{dt}=ik_n\left(u_{n+2}u^*_{n+1}-\frac{1}{4}u_{n+1}u^*_{n-1}+\frac{1}{8}u_{n-1}u_{n-2}\right)-\nu k_n^2 u_n+f_n, \tag{24}$$

where ν is the viscosity and f_n is a forcing term restricted to the first two shells.

The QL stream function (12) can be written in terms of u_n by taking $\phi_{n,1}=\mathrm{Re}(u_n)$, $\phi_{n,2}=\mathrm{Im}(u_n)$. The scaling exponents for the Eulerian structure functions ζ_p are computed numerically and listed in Table I. In Fig. 9 we report the statistics of Lagrangian doubling times compensated for by the theoretical scaling (23). Also in this case there is evidence of anomalous scaling in agreement with the multifractal prediction, confirming our previous findings with the stochastic velocity field. This result indicates that the relative dispersion statistics is not very sensitive to the details of the time dependence of the Eulerian velocity field [6,7].

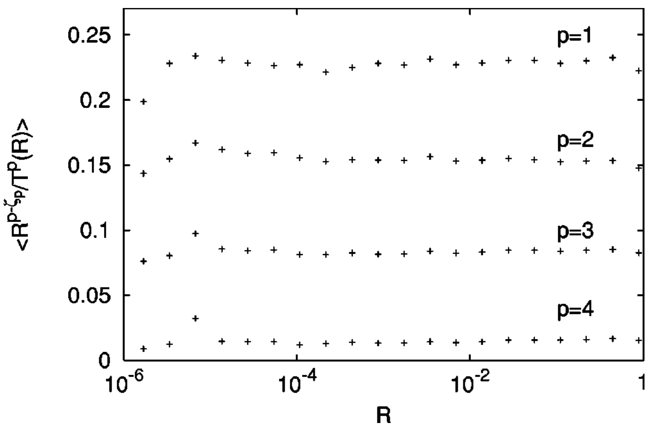

FIG. 9. Inverse time statistics $\langle 1/T^p(R)\rangle$ compensated for by the multifractal prediction R^{ζ_p-p} for the shell model simulation with $N=24$ shells, $\nu=10^{-8}$, and $k_0=0.05$. The average is over 10^4 realizations of particle pairs.

VII. CONCLUSIONS

In this paper we have proposed a simple and efficient method for generating a time dependent, turbulentlike velocity difference field in quasi-Lagrangian coordinates. The synthetic flow is constructed with prescribed two-point scaling properties (structure functions), thus allowing extensive investigations of the effects of intermittency on Lagrangian pair dispersion.

For non intermittent Kolmogorov-like turbulence, we find that our simulations agree with the original Richardson approach based on a diffusion equation for relative separation. In the case of intermittent turbulence we have found that the relative dispersion displays anomalous scaling exponents, and can no longer be described as a self-similar process. Relative dispersion intermittency can be included by a natural extension of the multifractal formalism to Lagrangian quantities. The predictions so obtained are in quite good agreement with our numerical simulations.

To better analyze numerical data, we have suggested an approach based on Lagrangian doubling times which leads to wider scaling ranges than the usual fixed time statistics, and thus it is very promising for data analysis. The present work is a first step toward the clarification of the Lagrangian-Eulerian relationship in fully developed turbulence. It would be extremely interesting to check our claims by mean of direct numerical simulations or laboratory experiments.

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[1] G.K. Batchelor, Proc. Cambridge Philos. Soc. **48**, 345 (1952).
[2] A. Monin and A. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975), Vol. 2.
[3] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1995).
[4] L.F. Richardson, Proc. R. Soc. London, Ser. A **110**, 709 (1926).
[5] N. Zovari and A. Babiano, Physica D **76**, 318 (1994).
[6] F.W. Elliott, Jr. and A.J. Majda, Phys. Fluids **8**, 1052 (1996).
[7] J.C.H. Fung and J.C. Vassilicos, Phys. Rev. E **57**, 1677 (1998).
[8] W.J. Cocke, Phys. Fluids **12**, 2488 (1969).
[9] S.A. Orszag, Phys. Fluids **13**, 2203 (1970).

[10] M.C. Jullien, J. Paret, and P. Tabeling, Phys. Rev. Lett. **82**, 2872 (1999).
[11] V.S. L'vov, E. Podivilov, and I. Procaccia, Phys. Rev. E **55**, 7030 (1997).
[12] H.A. Rose and P.L. Sulem, J. Phys. (France) **39**, 441 (1978).
[13] L. Biferale, G. Boffetta, A. Celani, A. Crisanti, and A. Vulpiani, Phys. Rev. E **57**, R6261 (1998).
[14] F. Anselmet, Y. Gagne, E.J. Hopfinger, and R.A. Antonia, J. Fluid Mech. **140**, 63 (1984).
[15] J.C.H. Fung, J.C.R. Hunt, N.A. Malik, and R.J. Perkins, J. Fluid Mech. **236**, 281 (1992).
[16] E. Aurell, G. Boffetta, A. Crisanti, G. Paladin, and A. Vulpiani, Phys. Rev. Lett. **77**, 1262 (1996).
[17] V. Artale, G. Boffetta, A. Celani, M. Cencini, and A. Vulpiani, Phys. Fluids A **9**, 3162 (1997).
[18] E.A. Novikov, Phys. Fluids A **1**, 326 (1989).
[19] A. Crisanti, G. Paladin, and A. Vulpiani, Phys. Lett. A **126**, 120 (1987).
[20] S. Grossmann and I. Procaccia, Phys. Rev. A **29**, 1358 (1984).
[21] M.F. Shlesinger, B.J. West, and J. Klafter, Phys. Rev. Lett. **58**, 1100 (1987).
[22] T. Bohr, M.H. Jensen, G. Paladin, and A. Vulpiani, *Dynamical Systems Approach to Turbulence* (Cambridge University Press, Cambridge, 1998).
[23] G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).
[24] L. Biferale, G. Boffetta, A. Celani, and F. Toschi, Physica D **127**, 187 (1999).
[25] V.S. L'vov, E. Podivilov, A. Pomyalov, I. Procaccia, and D. Vandembroucq, Phys. Rev. E **58**, 1811 (1998).